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SPECTRA OF F-SUM AND F-PRODUCT OF GRAPHS

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Abstract. Several graph operations based on subdivision graph and its variants have been introduced by many researchers and their spectral properties have been studied. The F-sum and F-product of graphs are one among these graph operations. In literature, many topological indices of F-sum and F-product of graphs have been examined. In this paper, we first introduce two matrix forms named as F-sum matrix and F-product matrix and describe their spectra, and then using the spectra of these matrices, we compute the spectra of F-sum and F-product of graphs.

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1. Introduction

All graphs considered in this paper are simple, that is, graphs with no loops and multiple edges. For a graph G with vertex labels $v_1, v_2, \ldots, v_n$, the adjacency matrix of G, denoted by $A(G)$, is the symmetric $(0,1)$-matrix of order n whose ijth entry is 1 if v_i and v_j are adjacent in G and 0 otherwise. In what follows, the collection of eigenvalues of $A(G)$ is known as the spectrum of G. There is an extensive literature available on the adjacency matrix of a graph and voluminous works carried out on this connectivity matrix reveal interesting connections between the graph structure and its spectrum, see [7, 8]. Graph operations play an important role in the theory of graph spectra. They provide counterexamples for several interesting families of graphs and also give rise to nice families of graphs. One of the interesting problem in spectral graph theory is to determine the spectra of graphs or graph operations. In literature, various graph operations like line graphs, join, disjoint union, NEPS of graphs, subdivision graph, corona product and its variants, etc. have been studied and their spectra are described in [1, 2, 5, 7, 8, 11, 12, 19].

The subdivision $S(G)$ of a graph G is the graph obtained by inserting an additional vertex into every edge of G. Let $Q(G)$ be the graph obtained from G by inserting a new vertex in each edge of G and then joining pairs of these new vertices by edges whenever the corresponding pairs of edges are incident in G. The graph $R(G)$ is defined as the graph obtained from G by introducing a new vertex corresponding to each edge of G and then joining each new vertex to the end vertices of the edge corresponding to it. The total graph of G, denoted by $T(G)$, is the graph whose set of vertices is the union of the set of vertices and set of edges of G, and two vertices of $T(G)$

are adjacent if and only if the corresponding elements of G are adjacent or incident.

Example 1.1. *S, Q, R and T graph of G.*

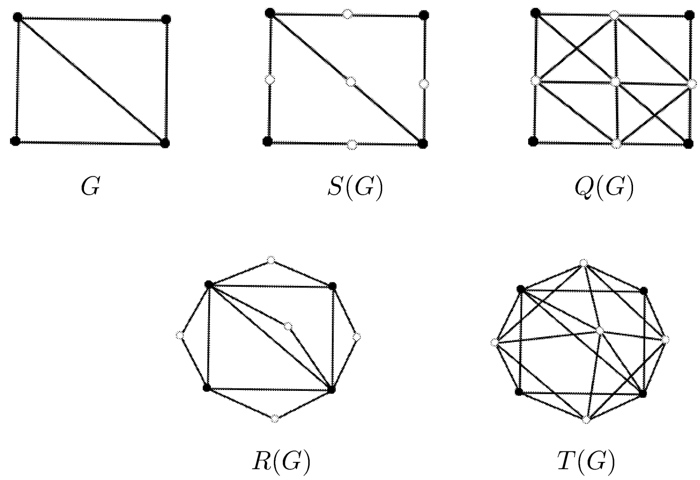
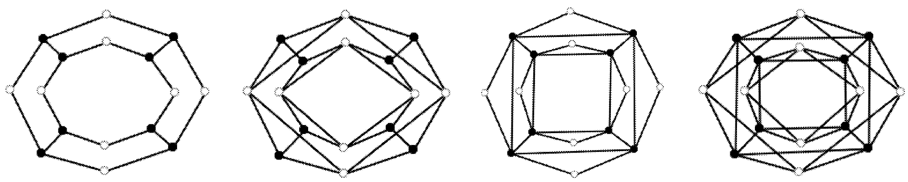


FIGURE 1. Graphs $S(G)$, $Q(G)$, $R(G)$ and $T(G)$.

In literature, many graph operations based on $S(G)$, $Q(G)$, $R(G)$ and $T(G)$ such as the subdivision-vertex neighborhood corona, subdivision-edge neighborhood corona, R-vertex neighborhood corona, R-edge neighborhood corona, subdivision-vertex join and subdivision-edge join of two graphs, Q-graph double corona, R-graph double corona, total double corona, subdivision double neighbourhood corona, Q-graph double neighbourhood corona, R-graph double neighbourhood corona and total double neighbourhood corona, etc. have been introduced. For their definitions and spectral properties see [6, 13, 15, 16].

In [10], Eliasi and Taeri based on $F(G)$ ($F = \{S, Q, R, T\}$) introduced four new sums of graphs called the F-sum of graphs. These are defined as follows: Suppose G and H are two graphs with vertex sets $V(G)$ and $V(H)$, and edge sets $E(G)$ and $E(H)$, respectively. Then the F-sum of G and H, denoted by $G +_F H$, is the graph with vertex set $V(G +_F H) = (V(G) \cup E(G)) \times V(H)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G +_F H$ are adjacent if and only if $u_1 = v_1 \in V(G)$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(F(G))$ (see Fig. 2). Eliasi and Taeri [10] also studied Wiener index of these graph operations. Recently, many researchers extended this study to other topological indices. In [17], Metsidik et al. determined the hyper-and reverse-Wiener indices of F-sum of graphs. Li and Wang [14] gave explicit expressions for the vertex PI indices of F-sum of graphs in terms of other indices of parent graphs. Likewise, An et al. [4] considered F-sum of graphs and gave two upper bounds for their degree distance index. In [3], Akhter et al. determined closed formulas for the F-index of these four graph operations and in [9], the Zagreb indices of these four operations on graphs have been studied by Deng et al.

Example 1.2. *Consider the graphs* $G = C_4$*, the cycle of length* 4*, and* $H = P_2$*, the path of length* 1*. Fig.* 2 *describes the* F*-sum* $G +_F H$*.*

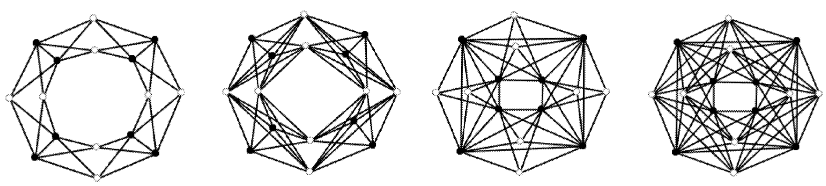


$C_4 +_S P_2$ $C_4 +_Q P_2$ $C_4 +_R P_2$ $C_4 +_T P_2$

FIGURE 2. Graphs $C_4 +_S P_2$, $C_4 +_Q P_2$, $C_4 +_R P_2$ and $C_4 +_T P_2$.

In analogous to F-sum of graphs, Sarala et al. [18] introduced four new graph operations called F-product of graphs which are defined as follows: The F-product of G and H, denoted by $G \times_F H$, is the graph with set of vertices $V(G +_F H) = (V(G) \cup E(G)) \times V(H)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G \times_F H$ are adjacent if and only if $u_1 = v_1 \in V(G)$ and $u_2v_2 \in E(H)$, or $u_1v_1 \in E(F(G))$ (see Fig. 3).

Example 1.3. *Consider the graphs* $G = C_4$*, the cycle of length* 4*, and* $H = P_2$*, the path of length* 1*. Fig.* 3 *describes the* F*-product* $G \times_F H$*.*



$C_4 \times_S P_2$ $C_4 \times_Q P_2$ $C_4 \times_R P_2$ $C_4 \times_T P_2$

FIGURE 3. Graphs $C_4 \times_S P_2$, $C_4 \times_Q P_2$, $C_4 \times_R P_2$ and $C_4 \times_T P_2$.

Sarala et al. also studied Zagreb indices of these graph operations. Motivated by all these works, in this paper, we study spectral properties of F-sum and F-product of graphs. We first define two matrix forms named as F-sum matrix and F-product matrix, and then describe their spectra. Next using the spectrum of F-sum matrix, we obtain the spectrum of the F-sum $G +_F H$ when G is regular, and using the spectrum of F-product matrix, we obtain the spectrum of the F-product $G \times_F H$ when both G and H are regular graphs.

2. Spectrum of F-sum matrix and F-product matrix

Let $M = (m_{ij})$ be an $n \times m$ matrix and N be an $p \times q$ matrix. Then the Kronecker product $M \otimes N$ of M and N is the $np \times mq$ matrix obtained by replacing each entry m_{ij} of M by $m_{ij}N$. It is well-known that $(M \otimes N)(R \otimes S) = MR \otimes NS$, whenever the products MR, NS are defined, and $\lambda\mu$ is the eigenvalue of $M \otimes N$, whenever λ and μ are the eigenvalues of the sqaure matrices M and N, respectively. Using the definition of Kronecker product, we now define F-sum matrix and F-product matrix as follows.

Let A, C, and D be real symmetric matrices of order m, n $(m \geq n)$ and r, respectively. Let B be a $m \times n$ real matrix. Consider the following two real square matrices of order $r(m+n)$.

$$F_s = \begin{bmatrix} A \otimes I_r & B \otimes I_r \\ B^T \otimes I_r & (I_n \otimes D) + (C \otimes I_r) \end{bmatrix}$$

and

$$F_p = \begin{bmatrix} A \otimes J_r & B \otimes J_r \\ B^T \otimes J_r & (I_n \otimes D) + (C \otimes J_r) \end{bmatrix}.$$

where I_r is the identity matrix of order r and J_r is the square matrix of order r whose all entries are 1. We call the matrix F_s as the F-sum matrix if F_s satisfies the conditions (a) and (b) stated below, and we call the matrix F_p as the F-product matrix if they satisfy all the following conditions.

(a) If X_i and Y_i are the singular vector pairs corresponding to singular values b_i of B for $i = 1, 2, \dots, n$, then X_i and Y_i are orthogonal eigenvectors of A and C, respectively, or equivalently, if $BY_i = b_iX_i$ and $B^TX_i = b_iY_i$ for $i = 1, 2, \dots, n$, then $AX_i = a_iX_i$ and $CY_i = c_iY_i$, where a_i and c_i are the eigenvalues of A and C, respectively.

(b) If $B^TX_i = 0$ for $i = n+1, n+2, \dots, m$, then X_i are orthogonal eigenvectors of A, that is, $AX_i = a_iX_i$ for $i = n+1, n+2, \dots, m$, where $a_i's$ are eigenvalues of A.

(c) D is a regular matrix, i.e., $D\mathbf{1}_r = d\mathbf{1}_r$, where $\mathbf{1_r}$ is a column vector of size r whose all entries are 1.

Let $d_1, d_2, \dots, d_r$ be the eigenvalues of D. In the following theorem, we list all the eigenvalues and eigenvectors of the F-sum matrix.

Theorem 2.1. *Let F_s be a F-sum matrix as defined above. Then the spectrum of F_s consists of the following:*

(i) $\dfrac{a_i + c_i + d_j \pm \sqrt{(a_i - c_i - d_j)^2 + 4b_i^2}}{2}$ *for* $b_i \neq 0$, $i = 1, 2, \dots, n$ *and* $j = 1, 2, \dots, r$;

(ii) $c_i + d_j$ *and* a_i *for* $b_i = 0, i = 1, 2, \dots, n$ *and* $j = 1, 2, \dots, r$;

(iii) a_i *repeating* r *times for* $i = n+1, n+2, \dots, m$.

Proof. Let $Z_1, Z_2, \dots, Z_r$ be orthogonal eigenvectors of D corresponding to the eigenvalues $d_1, d_2, \dots, d_r$ and let $\delta_{ij} \neq 0$ be any scalar. Then for $i =$

$1,2,\ldots,n$ and $j=1,2,\ldots,r$, we have

$$\begin{bmatrix} A\otimes I_r & B\otimes I_r \\ B^T\otimes I_r & (I_n\otimes D)+(C\otimes I_r)\end{bmatrix}\begin{bmatrix} X_i\otimes Z_j \\ \delta_{ij}Y_i\otimes Z_j\end{bmatrix}=\begin{bmatrix}(a_i+\delta_{ij}b_i)X_i\otimes Z_j \\ (b_i+\delta_{ij}(c_i+d_j))Y_i\otimes Z_j\end{bmatrix}.$$

The above matrix equation implies that the column vector $\begin{bmatrix} X_i\otimes Z_j \\ \delta_{ij}Y_i\otimes Z_j\end{bmatrix}$ is an eigenvector of F_s with corresponding eigenvalue $a_i+\delta_{ij}b_i$ if and only if $a_i+\delta_{ij}b_i=\dfrac{b_i}{\delta_{ij}}+c_i+d_j$, that is, $\begin{bmatrix} X_i\otimes Z_j \\ \delta_{ij}Y_i\otimes Z_j\end{bmatrix}$ is an eigenvector of F_s with corresponding eigenvalue $a_i+\delta_{ij}b_i$ if and only if $b_i\neq 0$ and $\delta_{ij}=\dfrac{c_i+d_j-a_i\pm\sqrt{(c_i+d_j-a_i)^2+4b_i^2}}{2b_i}$. This proves (i).

If $b_i=0$ for some $i=1,2,\ldots,n$, then for $j=1,2,\ldots,r$, we have

$$(1)\qquad \begin{bmatrix} A\otimes I_r & B\otimes I_r \\ B^T\otimes I_r & (I_n\otimes D)+(C\otimes I_r)\end{bmatrix}\begin{bmatrix} X_i\otimes Z_j \\ \mathbf{0}\end{bmatrix}=a_i\begin{bmatrix} X_i\otimes Z_j \\ \mathbf{0}\end{bmatrix}$$

and

$$\begin{bmatrix} A\otimes I_r & B\otimes I_r \\ B^T\otimes I_r & (I_n\otimes D)+(C\otimes I_r)\end{bmatrix}\begin{bmatrix} \mathbf{0} \\ Y_i\otimes Z_j\end{bmatrix}=(c_i+d_j)\begin{bmatrix} \mathbf{0} \\ Y_i\otimes Z_j\end{bmatrix}.$$

Moreover the matrix equation (1) is also true for $i=n+1,n+2,\ldots,m$. This proves (ii) and (iii). Thus we have listed all the eigenvalues of the F-sum matrix F_s together with their corresponding eigenvectors. □

Our next theorem describes the spectrum of F-product matrix.

Theorem 2.2. *Let F_p be a F-product matrix as defined above. Then the spectrum of F_p consists of the following:*

(i) 0 repeating $m(r-1)$ times;
(ii) d_j repeating n times for $j=2,3,\ldots,r$;
(iii) $\dfrac{a_ir+c_ir+d\pm\sqrt{(c_ir+d-a_ir)^2+4b_i^2r^2}}{2}$ for $b_i\neq 0$, $i=1,2,\ldots,n$;
(iv) a_ir, $d+c_ir$ for $b_i=0$, $i=1,2,\ldots,n$ and a_ir for $i=n+1,n+2,\ldots,m$.

Proof. Let $Z_1=1_r,Z_2,\ldots,Z_r$ be orthogonal eigenvectors of D corresponding to the eigenvalues $d_1=d,d_2,\ldots,d_r$, respectively. Then for $i=1,2,\ldots,m$ and $j=2,3,\ldots,r$, we have

$$\begin{bmatrix} A\otimes J_r & B\otimes J_r \\ B^T\otimes J_r & (I_n\otimes D)+(C\otimes J_r)\end{bmatrix}\begin{bmatrix} X_i\otimes Z_j \\ \mathbf{0}\end{bmatrix}=0$$

and

$$\begin{bmatrix} A\otimes J_r & B\otimes J_r \\ B^T\otimes J_r & (I_n\otimes D)+(C\otimes J_r)\end{bmatrix}\begin{bmatrix} \mathbf{0} \\ Y_i\otimes Z_j\end{bmatrix}=d_j\begin{bmatrix} \mathbf{0} \\ Y_i\otimes Z_j\end{bmatrix}.$$

This proves (ii) and (iii). Now let $\delta_i\neq 0$ be any scalar. Then for $i=1,2,\ldots,n$, we have

$$\begin{bmatrix} A\otimes J_r & B\otimes J_r \\ B^T\otimes J_r & (I_n\otimes D)+(C\otimes J_r)\end{bmatrix}\begin{bmatrix} X_i\otimes \mathbf{1_r} \\ \delta_iY_i\otimes \mathbf{1_r}\end{bmatrix}=\begin{bmatrix} r(a_i+\delta_ib_i)X_i\otimes \mathbf{1_r} \\ (b_ir+\delta_i(d+c_ir))Y_i\otimes \mathbf{1_r}\end{bmatrix}.$$

The above matrix equation implies that $\begin{bmatrix} X_i \otimes \mathbf{1_r} \\ Y_i \otimes \mathbf{1_r} \end{bmatrix}$ is an eigenvector of F_p if and only if $r(a_i+\delta_i b_i) = \frac{rb_i}{\delta_i}+d+c_i r$, that is, $\begin{bmatrix} X_i \otimes \mathbf{1_r} \\ Y_i \otimes \mathbf{1_r} \end{bmatrix}$ is an eigenvector of F_p if and only if $b_i \neq 0$ and $\delta_i = \frac{(c_i r + d - a_i r) \pm \sqrt{(c_i r + d - a_i r)^2 + 4b_i^2 r^2}}{2b_i r}$.

If $b_i = 0$ for $i = 1, 2, \ldots, n$, then observe that

$$\begin{bmatrix} A \otimes J_r & B \otimes J_r \\ B^T \otimes J_r & (I_n \otimes D) + (C \otimes J_r) \end{bmatrix} \begin{bmatrix} X_i \otimes \mathbf{1_r} \\ \mathbf{0} \end{bmatrix} = a_i r \begin{bmatrix} X_i \otimes \mathbf{1_r} \\ \mathbf{0} \end{bmatrix} \tag{2}$$

and

$$\begin{bmatrix} A \otimes J_r & B \otimes J_r \\ B^T \otimes J_r & (I_n \otimes D) + (C \otimes J_r) \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ Y_i \otimes \mathbf{1_r} \end{bmatrix} = d + c_i r \begin{bmatrix} \mathbf{0} \\ Y_i \otimes \mathbf{1_r} \end{bmatrix}.$$

Moreover the matrix equation (2) is also true for $i = n+1, n+2, \ldots, m$. This proves (iii) and (iv). □

3. Spectra of F-sum and F-product of two graphs

Let G be an k-regular graph with n vertices and m edges, and let H be any arbitrary graph on r vertices. We denote the edge-vertex incidence matrix of G by $M(G)$ and the adjacency matrix of the line graph of G by $L(G)$. It is well-known that $M(G)M^T(G) = L(G) + 2I_m$ and $M^T(G)M(G) = A(G) + kI_n$. In this section, we list all the eigenvalues of the F-sum $G +_F H$ of G and H and also we describe the spectrum of the F-product $G +_F H$ of G and H, when H is an d-regular graph. In the sequel, we denote the eigenvalues of G by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and the eigenvalues of H by $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r$.

Theorem 3.1. *Let $F(G)$ be the subdivision graph of G. Then*

(a) the spectrum of F-sum $G +_F H$ of G and H consists of

- *(i)* $\frac{\mu_j \pm \sqrt{\mu_j^2 + 4(\lambda_i + k)}}{2}$, *$\lambda_i \neq -k$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, r$;*
- *(ii) μ_j repeating p times for $j = 1, 2, \ldots, r$ and 0 repeating $(m-n+p)r$ times, where p is the multiplicity of $-k$ as an eigenvalue of G.*

(b) If H is an d-regular graph, then the spectrum of F-product $G \times_F H$ of G and H consists of

- *(i) 0 repeating (mr-n+p) times and d repeating p times, where p is the multiplicity of -k as an eigenvalue of G;*
- *(ii) μ_j repeating n times for $j = 2, 3, \ldots, r$;*
- *(iii)* $\frac{d \pm \sqrt{d^2 + 4(\lambda_i + k)r^2}}{2}$ *for $\lambda_i \neq -k$, $i = 1, 2, \ldots, n$.*

Proof. By proper labeling of the vertices of $G +_F H$ and $G \times_F H$, one can write down the adjacency matrices of $G +_F H$ and $G \times_F H$ as follows:

$$A(G +_F H) = \begin{bmatrix} \mathbf{0} & M(G) \otimes I_r \\ M^T(G) \otimes I_r & I_n \otimes A(H) \end{bmatrix},$$

$$A(G \times_F H) = \begin{bmatrix} \mathbf{0} & M(G) \otimes J_r \\ M^T(G) \otimes J_r & I_n \otimes A(H) \end{bmatrix}.$$

Now, comparing above matrices with the F-sum and F-product matrices, we have $A = \mathbf{0}$, $C = \mathbf{0}$, $B = M(G)$, $D = A(H)$ and $M^T(G)M(G) = kI_n + A(G)$. Thus $a_i = 0$, $i = 1, 2, \ldots, m$ and $c_i = 0$, $i = 1, 2, \ldots, n$, and $b_i^2 = \lambda_i + k$. Substituting these values in Theorem 2.1 and Theorem 2.2, we obtain the desired the result. □

Theorem 3.2. *Let F(G)=Q(G). Then*

(a) *the spectrum of F-sum $G +_F H$ of G and H consists of*

(i) $\dfrac{\lambda_i + \mu_j + k - 2 \pm \sqrt{(\lambda_i - \mu_j + k - 2)^2 + 4(\lambda_i + k)}}{2}$, $\lambda_i \neq -k$, $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, r$;

(ii) μ_j *repeating* p *times for* $j = 1, 2, \ldots, r$ *and* -2 *repeating* $(m-n+p)r$ *times, where* p *is the multiplicity of* $-k$ *as an eigenvalue of* G.

(b) *If H is an d-regular graph, then the spectrum of F-product $G \times_F H$ of G and H consists of*

(i) *0 repeating* $m(r-1)$ *times;*

(ii) μ_j *repeating* n *times for* $j = 2, 3, \ldots, r$;

(iii) $\dfrac{(\lambda_i + k - 2)r + d \pm \sqrt{(d - (\lambda_i + k - 2)r)^2 + 4(\lambda_i + k)r^2}}{2}$ *for* $\lambda_i \neq -k$, $i = 1, 2, \ldots, n$;

(iv) $-2r$ *repeating* $(m - n + p)$ *times and* d *repeating* p *times, where* p *is the multiplicity of* $-k$ *as an eigenvalue of* G.

Proof. We label the vertices of G in such a manner that the adjacency matrices of $G +_F H$ and $G \times_F H$ are

$$A(G +_F H) = \begin{bmatrix} L(G) \otimes I_r & M(G) \otimes I_r \\ M^T(G) \otimes I_r & I_n \otimes A(H) \end{bmatrix},$$

$$A(G \times_F H) = \begin{bmatrix} L(G) \otimes J_r & M(G) \otimes J_r \\ M^T(G) \otimes J_r & I_n \otimes A(H) \end{bmatrix}.$$

Now, comparing above matrices with the F-sum and F-product matrices, we have $A = L(G)$, $C = \mathbf{0}$, $B = M(G)$, $D = A(H)$ and $M^T(G)M(G) = kI_n + A(G)$. Thus

$a_i = \lambda_i + k - 2$ for $i = 1, 2, \ldots, n$ and $a_i = -2$ for $i = n+1, n+2, \ldots, m$, $c_i = 0$, $i = 1, 2, \ldots, n$, and $b_i^2 = \lambda_i + k$.

Substituting these values in Theorem 2.1 and Theorem 2.2, the result follows. □

Theorem 3.3. *Let F(G)=R(G). Then*

(a) *the spectrum of F-sum $G +_F H$ of G and H consists of*

(i) $\dfrac{\lambda_i + \mu_j \pm \sqrt{(\lambda_i + \mu_j)^2 + 4(\lambda_i + k)}}{2}$, $\lambda_i \neq -k$, $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, r$;

(ii) $\mu_j - k$ *repeating* p *times for* $j = 1, 2, \ldots, r$ *and* 0 *repeating* $(m - n + p)r$ *times, where* p *is the multiplicity of* $-k$ *as an eigenvalue of* G.

(b) *If H is an d-regular graph, then the spectrum of F-product $G \times_F H$ of G and H consists of*

(i) *0 repeating mr-n+p times and* $d-kr$ *repeating* p *times, where* p *is the multiplicity of* $-k$ *as an eigenvalue of* G;

(ii) μ_j *repeating* n *times for* $j=2,3,\ldots,r$;

(iii) $\dfrac{r\lambda_i+d\pm\sqrt{(d+r\lambda_i)^2+4(\lambda_i+k)r^2}}{2}$ *for* $\lambda_i\neq-k$, $i=1,2,\ldots,n$.

Proof. The adjacency matrices of $G+_F H$ and $G\times_F H$ can be formulated as

$$A(G+_F H)=\begin{bmatrix}\mathbf{0} & M(G)\otimes I_r\\ M^T(G)\otimes I_r & (I_n\otimes A(H))+(A(G)\otimes I_r)\end{bmatrix},$$

$$A(G\times_F H)=\begin{bmatrix}\mathbf{0} & M(G)\otimes J_r\\ M^T(G)\otimes J_r & (I_n\otimes A(H))+(A(G)\otimes J_r)\end{bmatrix}.$$

Now, comparing above matrices with the F-sum and F-product matrices, we have $A=\mathbf{0}$, $C=A(G)$, $B=M(G)$, $D=A(H)$ and $M^T(G)M(G)=kI_n+A(G)$. Thus $a_i=0$, $i=1,2,\ldots,m$ and $c_i=\lambda_i$, $i=1,2,\ldots,n$, and $b_i^2=\lambda_i+k$. Substituting these values in Theorem 2.1 and Theorem 2.2, the result follows. □

Theorem 3.4. *Let* $F(G)$ *be the total graph of* G. *Then*

(a) *the spectrum of F-sum* $G+_F H$ *of* G *and* H *consists of*

(i) $\dfrac{2\lambda_i+k-2+\mu_j\pm\sqrt{(\mu_j-k+2)^2+4(\lambda_i+k)}}{2}$, $\lambda_i\neq-k$, $i=1,2,\ldots,n$ *and* $j=1,2,\ldots,r$;

(ii) μ_j-k *repeating* p *times for* $j=1,2,\ldots,r$ *and* -2 *repeating* $(m-n+p)r$ *times, where* p *is the multiplicity of* $-k$ *as an eigenvalue of* G.

(b) *If* H *is an d-regular graph, then the spectrum of F-product* $G\times_F H$ *of* G *and* H *consists of*

(i) *0 repeating m(r-1) times;*

(ii) μ_j *repeating* n *times for* $j=2,3,\ldots,r$;

(iii) $\dfrac{(2\lambda_i+k-2)r+d\pm\sqrt{(d-(k-2)r)^2+4(\lambda_i+k)r^2}}{2}$ *for* $\lambda_i\neq-k$, $i=1,2,\ldots,n$;

(iv) $-2r$ *repeating m-n+p times and* $d-kr$ *repeating* p *times, where* p *is the multiplicity of* $-k$ *as an eigenvalue of* G.

Proof. The adjacency matrix of $G+_F H$ and $G\times_F H$ can be expressed as

$$A(G+_F H)=\begin{bmatrix}L(G)\otimes I_r & M(G)\otimes I_r\\ M^T(G)\otimes I_r & (I_n\otimes A(H))+(A(G)\otimes I_r)\end{bmatrix},$$

$$A(G\times_F H)=\begin{bmatrix}L(G)\otimes J_r & M(G)\otimes J_r\\ M^T(G)\otimes J_r & (I_n\otimes A(H))+(A(G)\otimes J_r)\end{bmatrix}.$$

Now, comparing above matrices with the F-sum and F-product matrix, we have $A=L(G)$, $C=A(G)$, $B=M(G)$, $D=A(H)$ and $M^T(G)M(G)=kI_n+A(G)$. Thus

$a_i=\lambda_i+k-2$ for $i=1,2,\ldots,n$ and $a_i=-2$ for $i=n+1,n+2,\ldots,m$, $c_i=\lambda_i$, $i=1,2,\ldots,n$, and $b_i^2=\lambda_i+k$.

Substituting these values in Theorem 2.1 and Theorem 2.2, the result follows. □

4. Conclusion

Here we have computed the spectra (adjacency spectra) of certain F-sum graphs and F-product graphs. These results can be used to construct infinite families of cospectral (adjacency) graphs. Further using the spectra of F-sum matrix and F-product matrix it is also possible to compute the spectra of the Laplacian matrix and signless Laplacian matrix of F-sum and F-product of graphs when the constituent graphs are regular.

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